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THESIS

AN ANALYTICAL APPROACH TO ASSESSING THE
VULNERABILITY OF BOMB SHELTERS TO AERIAL
BOMBING AND ARTILLERY ATTACK

by

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17 Cosati Codes			18 Subject Terms (continue on reverse if necessary and identify by block number)
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19 Abstract (continue on reverse if necessary and identify by block number)
This study examines the vulnerability of bomb shelters to aerial bombing and artillery attack, by modelling the bomb dump, i.e., the area within which the bomb shelters are located, as a cellular target. The stochastic process of hitting the dump with aerial bombs or artillery shells is modelled using suitable probability distributions, depending on the scenarios. Two measures of effectiveness are used:

(1)   The expected proportion of damage to the bomb dump by n weapons (bombs or shells) that are directed at the bomb dump, and

(2)   The probability that the ith shelter will be damaged due to an attack by n weapons that are directed at the bomb dump.

A generalized shelter hardness and vulnerability model is derived. Conditions under which shelter probability of kill and dump expected proportion of damage coincide and are independent of the relative shelter values are discussed. The effectiveness of various defender's and attacker's strategies is considered.

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An Analytical Approach to Assessing the Vulnerability of Bomb Shelters to Aerial Bombing
and Artillery Attack

by

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MASTER OF SCIENCE IN OPERATIONS RESEARCH
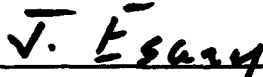
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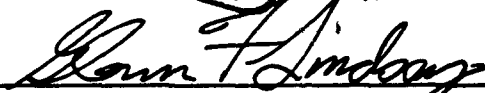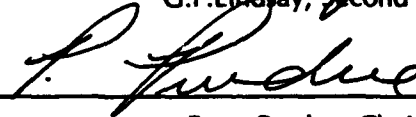NAVAL POSTGRADUATE SCHOOL
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Department of Operations Research

ABSTRACT

This study examines the vulnerability of bomb shelters to aerial bombing and artillery attack, by modelling the bomb dump, i.e., the area within which the bomb shelters are located, as a cellular target. The stochastic process of hitting the dump with aerial bombs or artillery shells is modelled using suitable probability distributions, depending on the scenarios. Two measures of effectiveness are used:

(1) The expected proportion of damage to the bomb dump by n weapons (bombs or shells) that are directed at the bomb dump, and

(2) The probability that the ith shelter will be damaged due to an attack by n weapons that are directed at the bomb dump.

A generalized shelter hardness and vulnerability model is derived. Conditions under which shelter probability of kill and dump expected proportion of damage coincide and are independent of the relative shelter values are discussed. The effectiveness of various defender's and attacker's strategies is considered.

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EXECUTIVE SUMMARY

This study examines the vulnerability of bomb shelters to aerial bombing and artillery attack by modelling the bomb dump, i.e., the area within which the bomb shelters are located, as a celluar target with m cells. The stochastic process of hitting the dump with aerial bombs or artillery shells is modelled using suitable hit probability distributions depending on the scenarios. Two measures of effectiveness are used:

(1)   The expected proportion of damage to the bomb dump by n weapons (bombs or shells) that are directed at the bomb dump, and

(2)   The probability that the ith shelter will be damaged due to an attack by n weapons that are directed at the bomb dump.

Two scenarios are used to examine the effectiveness of various strategies that can be adopted by a defending force or attacking force:

Scenario 1: High Level Aerial Bombing Using Large "Dumb" Bombs. Each aircraft will carry only one bomb. Each shelter is the same size, but is assigned an arbitrary weight (the proportion of the total value of the dump). Two subscenarios are examined, defined by the kill criteria or number of bombs to kill a shelter:

Kill Criteria 1:   One hit to kill a shelter, and

Kill Criteria 2: x hits to kill a shelter.

Since each aircraft releases only one bomb, each bomb will hit the dump independently with a probability p. Also, since high level bombing is adopted, it is assumed that each bomb is equally likely to hit any region in the dump.

Scenario 2: An Artillery Attack Scenario. Each shelter is the same size, but is assigned an arbitrary weight (the proportion of the total value of the dump). Since a shell is small in comparison to the size of a shelter, each artillery shell may only damage a small section of the shelter. Two kill criteria are examined:

Kill Criteria 1:   One hit on any section of a shelter
                    required to kill the shelter, and

Kill Criteria 2: x hits on the same section (any section)
                    of a shelter required to kill the shelter.

Each shell will hit the dump independently with a probability p and is equally likely to hit any region in the dump.

The models derived for Scenarios 1 and 2 lead to a generalized model which can be used for examining more sophisticated kill criteria based on an engineering analysis of shelter hardness and vulnerability to weapons.

It is shown that if a weapon that hits the dump is equally likely to hit any shelter in the dump and if the shelters are of the same size and have the same kill criteria, then two useful conclusions apply:

(1) The expected proportion of damage to the bomb dump and the probabilities of kill for the shelters do not depend on the weights assigned to the shelters, and

(2) The probabilities of kill for the shelters are the same, and are equal to the expected proportion of damage to the dump.

I INTRODUCTION

Weapons and ammunition are the jaws and claws of any fighting force. These high valued assets are housed in "bomb shelters". Bomb shelters are located in an area called a "bomb dump". Thus bomb dumps are often regarded as high value targets whose destruction will have a serious impact on the outcome of any conflict.

An attacking force will be concerned with what tactics to use, how many sorties to fly, or how many shells to fire to inflict a required level of damage. It will also be interested in how improved weapons accuracy will affect performance.

A defending force will be interested in knowing if survivability is improved by spacing out shelters over a larger area, or using harder shelters that are more difficult to defeat, or by building more shelters of smaller sizes.

This study addresses the above questions by examining two different scenarios using two measures of effectiveness:

(1) $E(D)$ - The expected proportion of damage to the dump by n weapons (bombs or shells) that are directed at the dump, and

(2) PK_i - The probability that a particular bomb shelter, the ith, will be damaged by n weapons that are directed at the dump.

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The two scenarios are:

Scenario 1: High Level aerial bombing using large "dumb" bombs. Each aircraft will carry only one bomb. Each shelter is the same size, but is assigned an arbitrary weight (the proportion of the total value of the dump). Two kill criteria are examined:

Kill Criteria 1: One hit to kill a shelter, and

Kill Criteria 2: x hits to kill a shelter.

Since each aircraft releases only one bomb, each bomb will hit the dump independently with a probability p. Also, as high level bombing is adopted, it is assumed that each bomb is equally likely to hit any region in the dump.

Scenario 2: An artillery attack scenario. Each shelter is the same size, but is assigned an arbitrary weight (the proportion of the total value of the dump). Since a shell is small in comparison to the size of a shelter, each artillery shell may only damage a small section of the shelter. Two kill criteria are examined:

Kill Criteria 1: One hit on any section of a shelter required to kill the shelter, and

Kill Criteria 2: x hits on the same section (any section) of a shelter required to kill the shelter.

Each shell will hit the dump independently with a probability p and is equally likely to hit any region in the dump.

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The scenarios examined are of course not exhaustive. However, the methodology used in modelling the dump and hit distributions can be used to examine other scenarios as the need arises.

The shelter kill criteria used in Scenarios 1 and 2 are based on simple assumptions about shelter size and vulnerability. A method for introducing more sophisticated kill criteria based on engineering analysis of shelter hardness and vulnerability to weapons directed against it is discussed in Chapter V.

Both scenarios assume that a weapon that hits the dump is equally likely to hit any region in the dump, and since the shelters are all the same size, equally likely to hit any shelter. The relative values of the shelter contents (weights w_i) may differ, but the kill criteria is the same for each shelter. The derivation of the expected proportion of damage, $E(D)$, to the dump and the probabilities, PK_i, that individual shelters are killed for the two scenarios leads to two useful observations about these and similar scenarios:

(1) $E(D)$ and the PK_i do not depend on the weights w_i assigned to the shelters, and

(2) Each PK_i is the same and equal to $E(D)$.

A basic hypothesis under which these observations apply is presented in Chapter V. The second observation can offer a convenient shortcut in the computation of $E(D)$, as illustrated in the analysis of Scenario 1, Kill Criteria 2.

Chapter II will describe bomb dumps and the weapons hit distributions.

II APPROACH

The area within which the bomb shelters are located is herein referred to as a bomb dump. The bomb dump is considered as a single target. This target is then divided into cells with each bomb shelter occupying one or more cells as shown in Figure 1. In essence, the bomb dump is modelled as a cellular target with the cells representing the bomb shelters or the empty spaces between them.

In each of the scenarios, n weapons are launched against the dump. Depending on the weapons used, each weapon will impact one or more cells if it hits the target. The number of weapons that hit the target is a random variable N with possible values $0,1,2,\ldots,n$.

Since the bomb shelters may be of differing values, each shelter is assigned an arbitrary weight w_i which is its proportion of the total value of the dump, $i=1,2,\ldots,s$, where s is the number of shelters. The empty spaces between the shelters are given zero weight. The total weight of all the shelters in a bomb dump will be 1.0.

The expected proportion of damage from k hits on the target is a random variable $D(k)$. The randomness is a result of randomness in the impact of weapons that hit the target and is a function of the shelter weights w_i and the probabilities

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$\delta_i(k)$ of the shelters being killed when k weapons hit the target.

The expected proportion of damage to the target $E(D)$ resulting from the n weapons that are launched against the target will range from 0 to 1. It will be a function of the random variables $D(k)$, $k = 0,1,2,\ldots,n$, and the hit probability distribution.
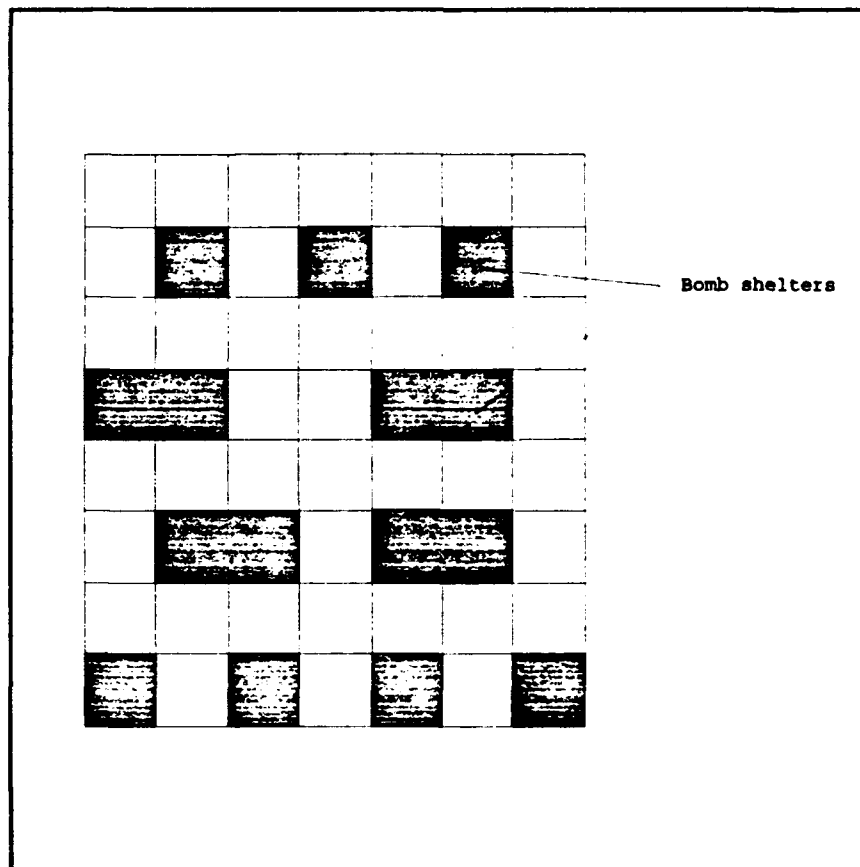

Figure 1:   Bomb Shelters modelled as cells in a target

Chapter III will illustrate how Scenario 1 is modelled and how the effectiveness of various defender's and attacker's strategies may be evaluated.

III SCENARIO 1

Scenario 1 is a high level bombing scenario in which the attacker uses a large "dumb" bomb. Each attacking aircraft will carry only one bomb. Each bomb will impact exactly one cell. Each shelter is assigned an arbitrary weight w_i. Two kill criteria are examined:

Kill Criteria 1:   One hit to kill a shelter, and

Kill Criteria 2: x hits to kill a shelter.

Since each aircraft releases only one bomb, each bomb will hit the dump independently with probability p. Also, as high level bombing is adopted, it is assumed that each hit is equally likely to impact any cell in the target. The target is divided into m cells of equal size.

A. KILL CRITERIA 1: ONE BOMB TO KILL A SHELTER

Since the bomb is equally likely to hit any cell in the target, the probability that shelter i will not be hit when a bomb lands on the target is

$$(1 - c_i/m) ,$$

where c_i is the number of cells that shelter i occupies.

Therefore the probability of at least one bomb hitting shelter i when k bombs have hit the target area is

$1 - (1 - c_i/m)^k.$

The probability of the *ith* shelter being killed when k bombs hit the target is

$$\delta_i(k) = 1 - (1 - c_i/m)^k. \quad \text{-----------------------} (1.1)$$

The proportion of damage from k hits is

$$D(K) = \sum_{i=1}^{s} \delta_i(k) w_i \ .$$

If each shelter occupies exactly one cell, then $\delta_i(k)$ is the same for each i ($\delta_1(k) = \delta_2(k) = \ldots = \delta_s(k) = \delta(k)$), and

$$D(K) = \delta_i(k) \sum_{i=1}^{s} w_i$$

$$= \delta(k), \quad \text{--------------------------------} (1.2)$$

and substituting for $\delta(k)$,

$$D(k) = 1 - (1 - 1/m)^K. \quad \text{--------------------------} (1.3)$$

Therefore the expected proportion of damage from the attack is

$$E(D) = \sum_{k=1}^{n} D(k) \, Pr(N=k)$$

$$= \sum_{k=1}^{n} \delta(k) \, Pr(N=k). \quad \text{------------------------(1.4)}$$

The hit distribution is binomial since each bomb hits the target independently with probability p, and

$$E(D) = \sum_{k=1}^{n} \left[1 - \left(1 - \frac{1}{m}\right)^{k} \right] \binom{n}{k} p^{k} (1-p)^{n-k}$$

$$= 1 - \sum_{k=1}^{n} \left[\left(1 - \frac{1}{m}\right) p \right]^{k} \binom{n}{k} (1-p)^{n-k}$$

$$= 1 - [(1-1/m)p + (1-p)]^{n}$$

$$= 1 - (1 - p/m)^{n}. \quad \text{------------------------(1.5)}$$

Also,

$$PK_{i} = \sum_{k=1}^{n} \delta_{i}(k) \, Pr[N=k]$$

$$= \sum_{k=1}^{n} \delta(k) \, Pr(N=k]. \quad \text{------------------------(1.6)}$$

As in Equations (1.4) and (1.5),

$$PK_{i} = 1 - (1 - p/m)^{n}. \quad \text{------------------------(1.7)}$$

From Equations (1.4) and (1.6) the probability of kill for shelter i is equal to the expected damage to the dump from n

bombs and is not dependent on the relative weight of each shelter w_i. This is true regardless of the hit probability distribution $Pr(N=k)$, $k=0,1,\ldots,n$. The only condition for the PK_i to be the same as the expected damage $E(D)$ is for the $\delta_i(k)$'s to be the same for each shelter i, and this will be true if each shelter is of the same size and a bomb that hits the target is equally likely to hit any shelter in the dump.

B. KILL CRITERIA 2: x BOMBS TO KILL A SHELTER

Again if each shelter occupies exactly one cell, then the probability of the ith shelter being hit when one bomb lands in the target area $= 1/m$.

Out of k hits on the target, the probability that the ith cell is hit by j bombs is

$$\binom{k}{j}\left[\frac{1}{m}\right]^{j}\left[1-\frac{1}{m}\right]^{k-j}.$$

Therefore,

$$\delta_i(k) \quad = \quad \text{Pr[shelter } i \text{ killed} | k \text{ hits on the dump]}$$

$$= \quad \text{Pr[out of } k \text{ hits on the dump, at least } x \text{ bombs land on shelter } i]$$

$$= \sum_{j=x}^{k}\binom{k}{j}\left[\frac{1}{m}\right]^{j}\left[1-\frac{1}{m}\right]^{k-j} = \delta(k). \quad \text{------------}(1.8)$$

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Therefore the proportion of damage when k bombs hit the dump is

$$D(k) = \sum_{i=1}^{s} w_i \delta_i(k)$$

$$= \delta(k) \sum_{i=1}^{s} w_i$$

$$= \delta(k). \quad \text{------------------------------} (1.9)$$

The expected proportion of damage is

$$E(D) = \sum_{k=0}^{n} D(k) Pr(N=k)$$

$$= \sum_{k=0}^{n} \delta(k) Pr(N=k), \quad \text{----------------------} (1.10)$$

and

$$PK_i = \sum_{k=1}^{n} \delta_i(k) Pr(N=k)$$

$$= \sum_{k=1}^{n} \delta(k) Pr(N=k). \quad \text{----------------------} (1.11)$$

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Assuming the hit probability to be binomial with probability p and substituting for $\delta(k)$, then

$$PK_i = E(D) = \sum_{k=0}^{n} \sum_{j=x}^{k} \binom{k}{j}\left[\frac{1}{m}\right]^j\left[1-\frac{1}{m}\right]^{k-j}\binom{n}{k}p^k(1-p)^{n-k}. \quad \text{--------}(1.12)$$

Again the expected damage and the probability of kill for shelter i are the same regardless of the hit distribution as shown by Equations (1.10) and (1.11), and independent of w_i. This will be true as long as the probability $\delta_i(k)$ of kill for shelter i given k hits on the target is the same for all shelters.

Equation (1.12) can be further simplified by combining the hit probability p on the dump and the hit probability $1/m$ on the shelter given a hit on the dump, as shown in Figure 2.

Since a bomb is equally likely to hit any shelter if it hits the dump, the probability of a hit on shelter i will then be p/m. Therefore

PK_i = Pr[out of n bombs, at least x bombs hit shelter i]

$$= \sum_{k=x}^{n} \binom{n}{k}\left[\frac{p}{m}\right]^k\left[1-\frac{p}{m}\right]^{n-k}. \quad \text{--------------------------}(1.13)$$

Since $E(D) = PK_i$,

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$$E(D) = PK_i = \sum_{k=x}^{n} \binom{n}{k} \left[\frac{p}{m}\right]^k \left[1 - \frac{p}{m}\right]^{n-k}. \quad \text{------------(1.14)}$$
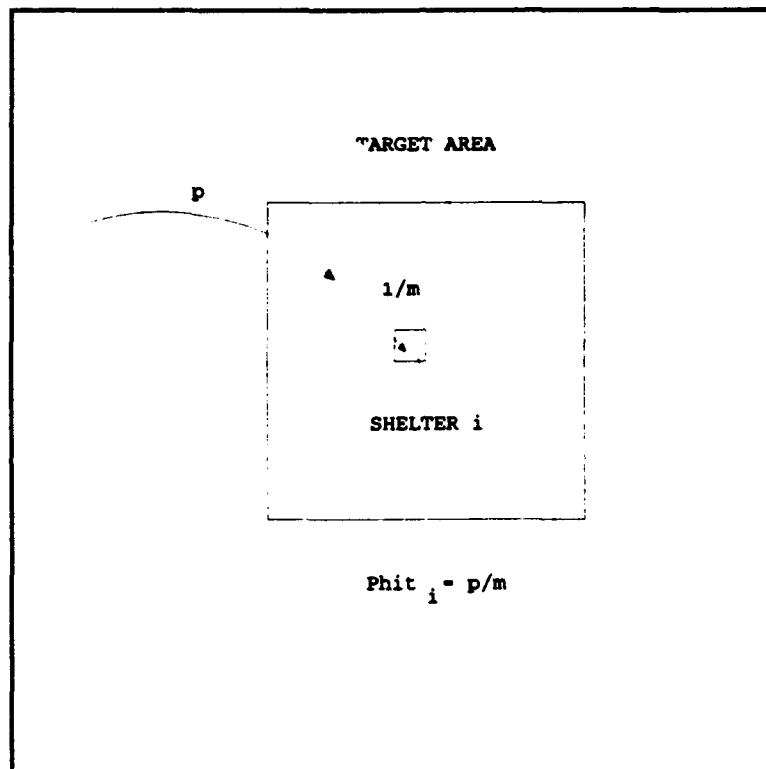
Figure 2: Probability of hit on shelter *i*

C. ANALYSIS OF DEFENDER'S STRATEGY

The defender's strategy may be to improve the survivability of the shelters by spreading the shelters over a larger area or hardening the shelters. The hardness of a

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shelter is defined by x, the number of hits required to kill the shelter. The spread of the shelters is defined by m, the number of cells in the target area (many of the cells will just be empty spaces between the shelters).

1. Hardening the Shelters

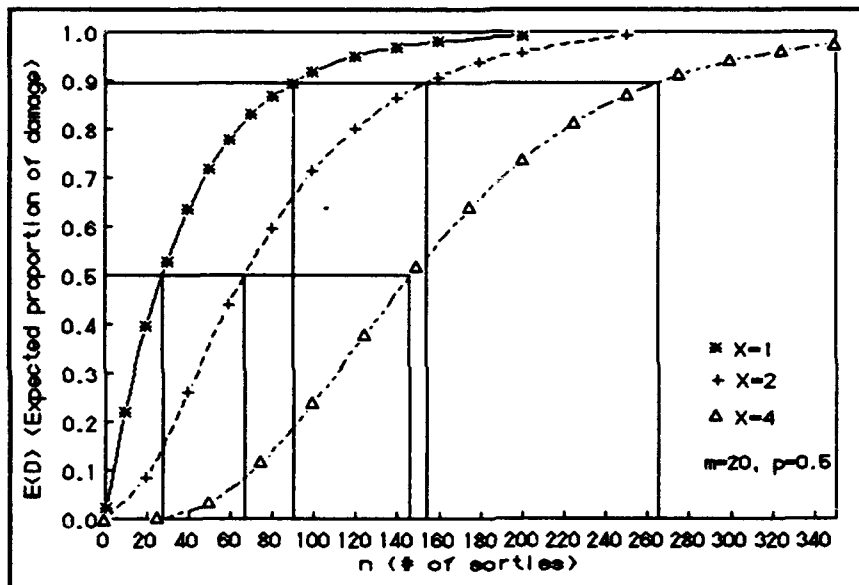Figure 3 shows the graph of $E(D)$ vs n for $m = 20$ and $p = 0.5$ and $x = 1,2$ or 4.

Figure 3: $E(D)$ vs n for shelters of different hardness, $p=0.5,m=20$

It shows how the expected damage is reduced as the hardness of the shelters is increased. From this figure, the increase in number of sorties required to achieve a given level of damage as the hardness of the shelters increases can be computed as shown in Table 1.

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Table 1: Number of sorties required to achieve a given level of expected damage, $E(D)$, against shelters of different hardness, x, if number of cells, $m=20$ and accuracy of weapon is $p=0.5$

$p=0.5, m=20$	$x = 1$	$x = 2$	$x = 4$
$E(D) = 0.5$	28	68	145
$E(D) = 0.9$	90	154	265

In Scenario 1, where $x=1$, $E(D)$ can be rewritten as

$$E(D|x=1) = \sum_{k=1}^{n} \binom{n}{k}\left[\frac{p}{m}\right]^k\left[1-\frac{p}{m}\right]^{n-k}, \quad \text{------------} (1.15)$$

which is in the same form as Equation (1.14) for arbitary x in Scenario 2. From Equations (1.14) and (1.15), the reduction in expected damage as the shelters are hardened can be computed from the expression

$$E(D|x=x) - E(D|x=1) = \sum_{k=1}^{x-1} \binom{n}{k}\left[\frac{p}{m}\right]^k\left[1-\frac{p}{m}\right]^{n-k}, \quad \text{------} (1.16)$$

In general, the reduction in expected damage as the shelters are hardened from x_1 to x_2 can be expressed as

$$E(D|x=x_1) - E(D|x=x_2) = \sum_{k=x_1}^{x_2-x_1} \binom{n}{k}\left[\frac{p}{m}\right]^k\left[1-\frac{p}{m}\right]^{n-k}. \quad \text{---} (1.17)$$

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2. Spreading the Shelters over a Larger Area

If the shelters are spread over a larger area, the number of cells *m* in the target area will increase as will the probability of hit on the target area, *p*.

Since expected damage is simply a function of *p/m*, spreading the shelters over a larger area will only improve the survivability if the new *p/m* is reduced. The amount of improvement in survivability will depend on how much *p/m* is reduced. If however *p/m* increases as a result of spreading the shelter over a larger area, then the survivability will drop! As a case study, let *p*=0.5 when *m* is 20 and *p*=0.9 when *m* is increased to 50. Figure 4 shows the *E*(*D*) plotted against the number of sorties *n*.
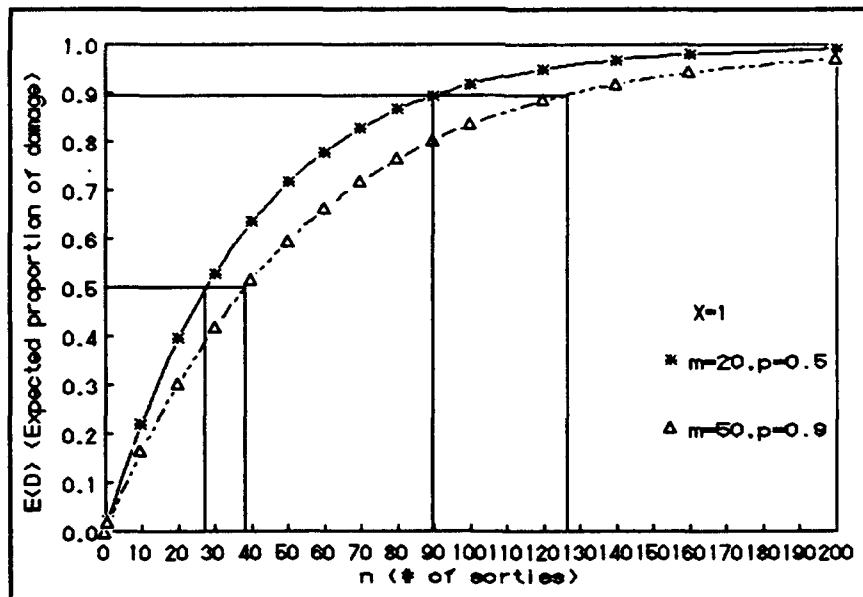

Figure 4:   *E*(*D*) vs *n* for different shelter spreads

It shows how the expected damage is reduced as the shelters are spread over a larger area. From this figure the increase in number of sorties required to achieve a given level of damage as the shelters are spread over a larger area can be computed as shown in Table 2

Table 2:   Number of sorties required to achieve a given level of expected damage, $E(D)$, against shelters of different shelter spreads as defined by the number of cells m and the hit probability p if the hardness of each shelter is $x=1$

$x=1$	$p=0.5, m=20$	$p=0.9, m=50$
$E(D) = 0.5$	28	68
$E(D) = 0.9$	90	154

3. Spread vs Hardening of Shelters

Since it is not possible to increase the hardness of the shelters infinitely and neither is it possible to spread the shelters over a infinitely large area, there will be upper bounds on x and m. As a case study, let $x=1$, $m=20$, and $p=0.5$. Two possible defender's strategies are examined:

(1)   Increase hardness of all shelters to $x=2$, and

(2)   Increase the shelter spread by 50% from $m=20$ to $m=30$ whereby in so doing, p increases from 0.5 to 0.6.

From Figure 5, increasing hardness to $x=2$ is a better option since $E(D)$ is lower for a given number of sorties if $p=0.6$ for $m=30$. Even if the probability of hit stays at 0.5

when the spread is increased to 30, the hardening option of x=2 is still better for E(D) less than 0.95. For E(D) greater then 0.95, the reverse is true though the difference is small. To achieve an E(D) of 0.95 requires an improbably massive strike of over 220 sorties. Thus for this case study, it can be concluded that hardening of shelters is a better option with regard to survivability.

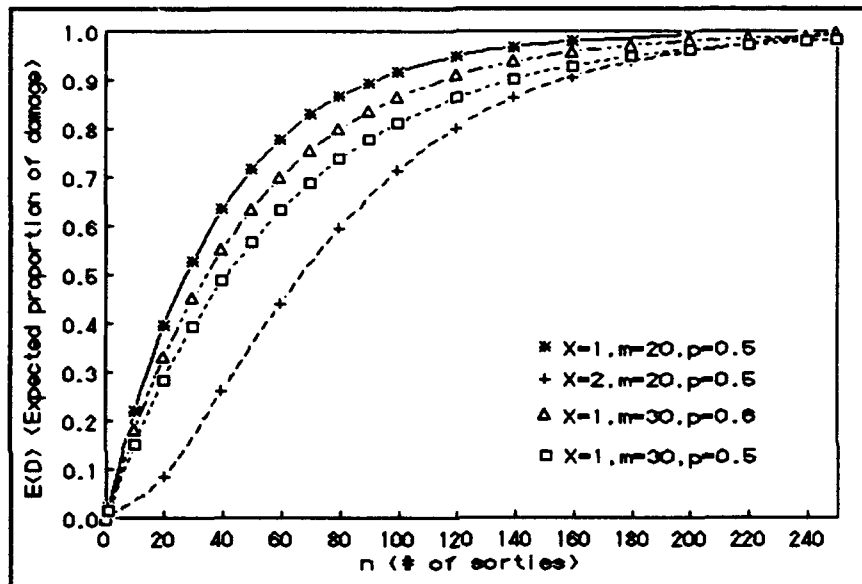Figure 5:   Comparison of increasing *m* vs increasing *x*

D. ANALYSIS OF ATTACKER'S STRATEGY

The attacker's interest will be to know what he will gain in terms of a reduced number of sorties by improving the accuracy of his bombs.

Figure 6 show how expected damage is increased as the accuracy of the weapons improves. From this figure the reduction in number of sorties required to achieve a given

level of damage as the accuracy of the weapons improves can be computed as shown in Table 3.

Table 3:   Number of sorties required to achieve a given level of expected damage, $E(D)$, when using weapons of different accuracy, p if the number of cells is $m = 20$ and the hardness of shelters is $x=2$

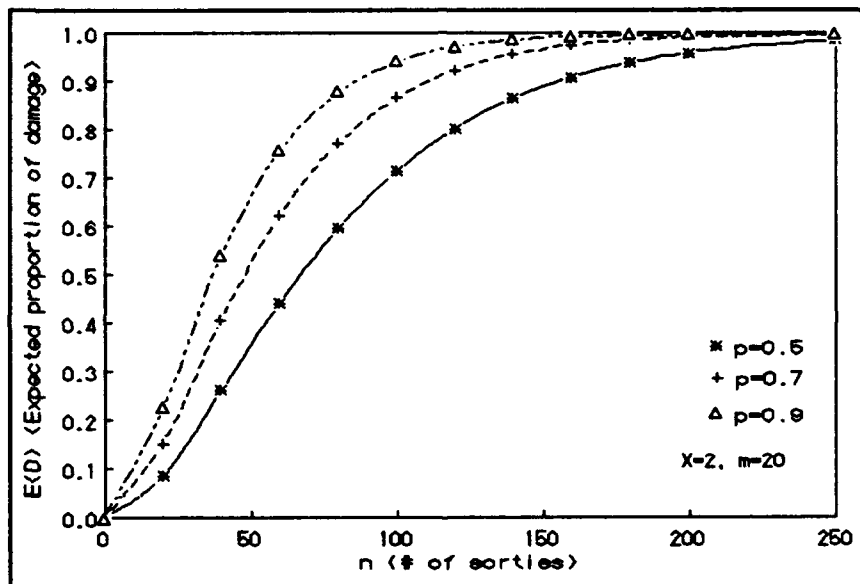| $m=20,x=2$	$p = 0.5$	$p = 0.7$	$p = 0.9$
$E(D) = 0.5$	67	49	38
$E(D) = 0.9$	152	110	87

Figure 6: $E(D)$ vs n for weapons of different accuracies

Figure 7 shows how the number of sorties required to achieve a given level of damage is reduced as accuracy increases.

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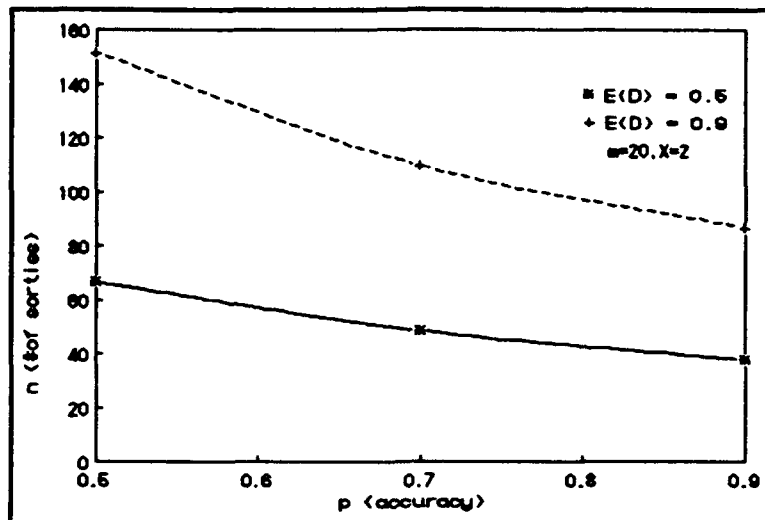
Figure 7: Graph of *p* vs # of sorties required to achieve a given level of damage

E. SUMMARY OF ANALYSIS

The defender's strategy may be to improve the survivability of the shelters through hardening the shelters or spreading the shelters over a wider area. The above discussions illustrate how to compute the effectiveness of each strategy in terms of the number of sorties required to achieve a given level of damage, as the shelters are hardened or spread over a larger area.

For the strategy of spreading the shelters over a larger area, survivability will only improve if the new *p/m* is reduced.

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The above discussions also illustrate how the option of hardening or spreading the shelters may be compared using the mathematical models derived.

From the attacker's point of view, the effectiveness of increasing weapons accuracy on the number of sorties required to achieve a given level of damage may be readily computed, as illustrated in the preceeding discussion.

In the same manner, Chapter IV will examine the artillery attack Scenario 2.

IV SCENARIO 2

Scenario 2 is an artillery attack scenario in which each shell will hit the target independently with a probability p and is equally likely to hit any region in the target area. Each shelter is assigned an arbitrary weight. Since the shell is small in comparison to the size of the shelter, each artillery shell will only damage a small section of the shelter. Two kill criteria are examined:

Kill Criteria 1:   One hit on any section of the shelter is sufficient to kill the shelter, and

Kill Criteria 2: x hits on the same section (any section) of the shelter are required to kill the shelter.

The bomb dump target is cellular with m cells. Each shelter is assumed to occupy one cell. Each cell is in turn divided into a subcells or subsections. For a target area, the number of possible subcells m times a is constant.

A. KILL CRITERIA 1: ONE SHELL HIT ON ANY SECTION TO KILL A SHELTER

For the first kill criteria, any hit on the shelter will kill the shelter. This is essentially the same as Scenario 1, under its Kill Criteria 1.

Therefore,

$$\delta_i(k) = \text{Pr[shelter } i \text{ killed} | k \text{ rounds hit the target]}$$

$$= 1 - (1 - 1/m)^k. \text{ -------------------------(2.1)}$$

The proportion of damage from k hits is

$$D(k) = \sum_{i=1}^{s} \delta_i(k) w_i.$$

Since $\delta_i(k) = \delta(k), \quad i = 1,\ldots,s$

$$D(k) = \delta(k) \sum_{i=1}^{s} w_i$$

$$= \delta(k). \text{ --------------------------------(2.2)}$$

The proportion of target damage is

$$E(D) = \sum_{k=1}^{n} D(k) \, Pr(N=k)$$

$$= \sum_{k=1}^{n} \delta(k) \, Pr(N=k). \text{ ------------------------(2.3)}$$

If the hit distribution is binomial with a hit probability of p, then

$$E(D) = 1 - (1 - p/m)^n, \text{ -------------------(2.4)}$$

and the probability that the *ith* shelter is killed is

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$$PK_i = \sum_{k=1}^{n} \delta_i(k) \, Pr(N{=}k) = E(D) = 1-(1-p/m)^n. \quad \text{------}(2.5)$$

Again from Equation 2.5, the probability of kill for shelter i is equal to the expected damage to the dump from n bombs and is not dependent on the relative weight of each shelter w_i. This is true regardless of the hit probability distribution $Pr(N{=}k)$, $k{=}0,1,\ldots,n$. The only condition for PK_i to be the same as the expected damage $E(D)$ is for the $\delta_i(k)$'s to be the same for each shelter i and this will be true if each shelter is of the same size and a bomb that hits the target is equally likely to hit any shelter in the dump.

B. KILL CRITERIA 2: x HITS ON ANY SUBCELL REQUIRED TO KILL A SHELTER

Now consider the case where Shelter i is killed if at least x shells hit one or more of its subcells.

If j shells hit a shelter, let $j_1,\ldots,j_a$ represent the numbers of these shells that hit subcells $1,\ldots,a$. Then $j_1+\ldots+j_a=j$. Let $K(x)$ be the set of vectors $j = \{j_1,\ldots j_a\}$ with at least one $j_l \geq x$, i.e. the set of assignments of shells to subcells for which the shelter is killed.

If k shells hit the target, the probability that j shells hit the ith shelter is

$$\binom{k}{j}\left(\frac{1}{m}\right)^j\left(1-\frac{1}{m}\right)^{k-j} \qquad j=0,1,\ldots,k.$$

The probability that the ith shelter is killed by j shells is

$$\beta_i(j) = \sum_{j \in K(x)} \frac{j!}{j_1!\ldots j_a!}\left(\frac{1}{a}\right)^{j_1}\cdots\left(\frac{1}{a}\right)^{j_a} = \sum_{j \in K(x)} \frac{j!}{j_1!\ldots j_a!}\left(\frac{1}{a}\right)^j. \qquad -(2.6)$$

The probability that the ith shelter is killed by k shells that hit the target is then

$$\delta_i(k) = \sum_{j=0}^{k} \beta_i(j)\binom{k}{j}\left(\frac{1}{m}\right)^j\left(1-\frac{1}{m}\right)^{k-j}. \qquad ----------------(2.7)$$

Since all of the shelters have the same number of subcells, then $\beta_1(j) = \ldots = \beta_s(j) = \beta(j)$, $j=0,\ldots,k$, and $\delta_1(k) = \ldots = \delta_s(k) = \delta(k)$, $k=0,\ldots,n$.

The proportion of damage from k hits on the target area is

$$D(k) = \sum_{i=1}^{s} w_i \delta_i(k) = \delta(k)\sum_{i=1}^{s} w_i = \delta(k).$$

The expected proportion of damage from n shells is

$$E(D) = \sum_{k=1}^{n} D(k)Pr(N=k) = \sum_{k=1}^{n} \delta(k)Pr(N=k), \qquad --------(2.8)$$

The probability of shelter i being killed is

$$PK_i = \sum_{k=1}^{n} \delta_i(k) Pr(N=k) = \sum_{k=1}^{n} \delta(k) Pr(N=k). \quad \text{--------}(2.9)$$

Therefore,

$$PK_i = E(D) = \sum_{k=0}^{n} \delta(k) Pr[N=k]$$

$$= \sum_{k=0}^{n} \sum_{j=0}^{k} \beta(j) \binom{k}{j}\left(\frac{1}{m}\right)^{j}\left(1-\frac{1}{m}\right)^{k-j}\binom{n}{k}p^{k}(1-p)^{n-k}.$$

Reversing the order of summation,

$$PK_i = E(D) = \sum_{j=0}^{n} \beta(j) \sum_{k=j}^{n} \binom{k}{j}\left(\frac{1}{m}\right)^{j}\left(1-\frac{1}{m}\right)^{k-j}\binom{n}{k}p^{k}(1-p)^{n-k}.$$

Let $r=k-j$. Then

$$PK_i = E(D) = \sum_{j=0}^{n} \beta(j) \sum_{k=j}^{n} \frac{k!}{j!\,(k-j)!}\left(\frac{1}{m}\right)^{j}\left(1-\frac{1}{m}\right)^{k-j} \frac{n!}{k!\,(n-k)!}p^{k}(1-p)^{n-k}$$

$$= \sum_{j=0}^{n} \beta(j) \frac{n!}{j!\,(n-j)!}\left(\frac{p}{m}\right)^{j} \sum_{k=j}^{n} \frac{(n-j)!}{(k-j)!\,(n-k)!}\left[(1-\frac{1}{m})p\right]^{k-j}(1-p)^{n-k}$$

$$= \sum_{j=0}^{n} \beta(j) \frac{n!}{j!\,(n-j)!}\left(\frac{p}{m}\right)^{j} \sum_{r=0}^{n-j} \frac{(n-j)!}{r!\,(n-j-r)!}\left[(1-\frac{1}{m})p\right]^{r}(1-p)^{n-j-r}$$

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$$= \sum_{j=0}^{n} \beta(j) \frac{n!}{j!(n-j)!} \left(\frac{p}{m}\right)^{j} \left[(1-\frac{1}{m})p + (1-p)\right]^{n-j}$$

$$= \sum_{j=0}^{n} \beta(j) \frac{n!}{j!(n-j)!} \left(\frac{p}{m}\right)^{j} \left(1-\frac{p}{m}\right)^{n-j}$$

$$= \sum_{j=0}^{n} \beta(j) \binom{n}{j} \left(\frac{p}{m}\right)^{j} \left(1-\frac{p}{m}\right)^{n-j}. \qquad \text{------------------------(2.10)}$$

Again the expected damage and the probability of kill for shelter i are the same regardless of the hit distribution as shown by Equation (2.10) and independent of w_i. This will be true as long as the probability $\delta_i(k)$ of kill for shelter i given k hits on the target is the same for all shelters.

C. ANALYSIS OF DEFENDER'S STRATEGY

The defender's strategy may be to improve the survivability of the shelters by spreading the shelters over a larger area or hardening the shelters or reducing the size of the shelters. The hardness of a shelter is defined by x, the number of hits that are required to kill the shelter. The spread of the shelters is defined by m, the number of cells in the target area (many of the cells will just be empty spaces between the shelters). The size of a shelter is defined by a.

1. Hardening the Shelters

Figure 8 shows how expected damage is reduced as the hardness of the shelters increases. If the shelters are hardened to take x hits, then the amount of reduction in expected damage is

$$\sum_{j=0}^{x-1} \beta_i(j) \binom{n}{j} \left[\frac{p}{m}\right]^j \left[1-\frac{p}{m}\right]^{n-j}.$$

From this figure the increase in the number of shells required to achieve a given level of damage as the hardness of the shelters increases can be computed as shown in Table 4.

Table 4: Number of shells required to achieve a given level of expected damage, $E(D)$, against shelters of different hardness, x when the weapon accuracy is $p=0.5$ and the shelter size is defined by the number of cells $m=29$ and the number of subcells $a=4$

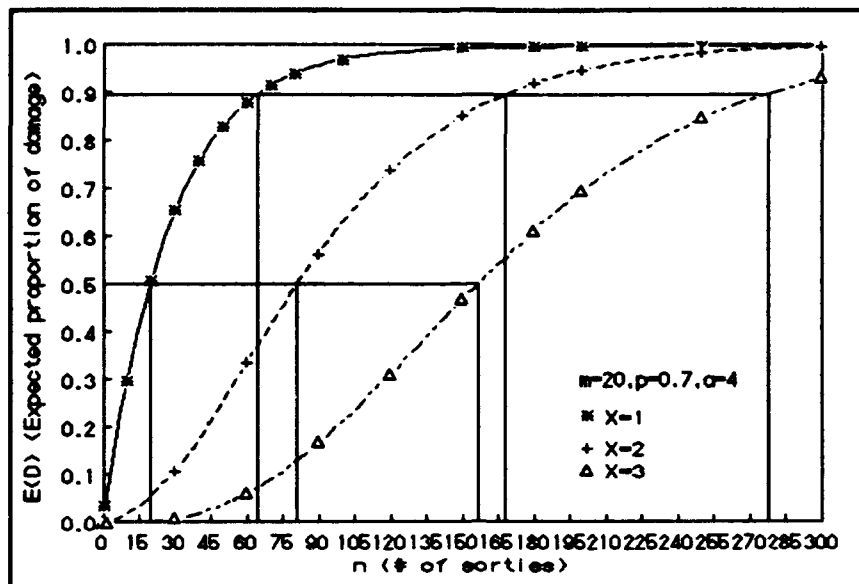| $p=0.5, m=20, a=4$	$x = 1$	$x = 2$	$x = 3$
$E(D) = 0.5$	20	81	158
$E(D) = 0.9$	65	168	278

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Figure 8: $E(D)$ vs n for shelters of different hardness

2. Spreading the Shelters over a Larger Area

If the shelters are spread over a larger area, m, the number of cells in the target area will increase as will the probability of a hit on the target area. If all the shelters are of the same size, then the expected damage is simply a function of p/m. Therefore spreading the shelters over a larger area will only improve survivability if p/m is reduced. The amount of improvement will depend on how much p/m is reduced. Again, if p/m increases as a result of spreading the shelters over a larger area, then survivability will drop!

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As a case study, Figure 9 shows a graph of $E(D)$ vs n for different shelter spreads as defined by $m=10,20,30$ with $p/m = 0.05$, 0.035 and 0.03. As expected, $E(D)$ is smallest for the smallest p/m.
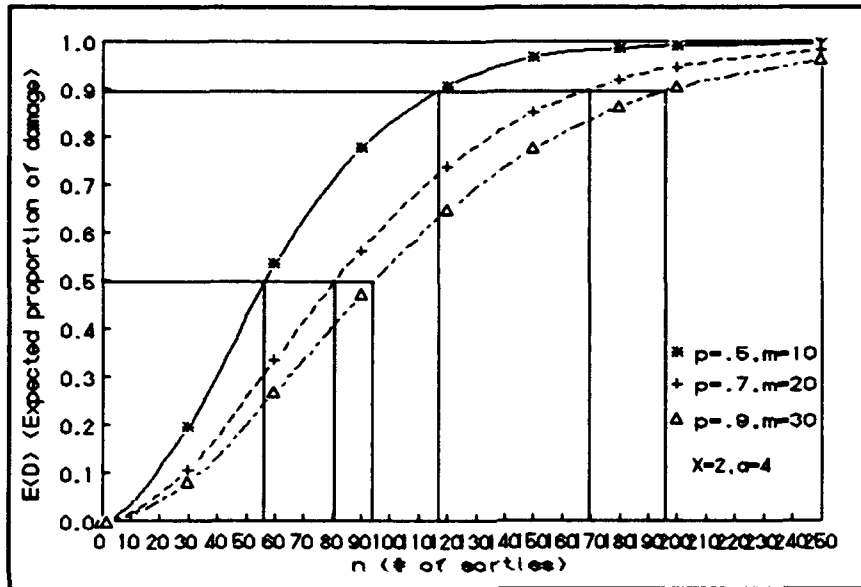

Figure 9: $E(D)$ vs n for shelters of different spread

From this figure the increase in number of shells required to achieve a given level of damage as the hardness of the shelters increases is computed as shown in Table 5.

Table 5: Number of shells required to achieve a given level of expected damage, $E(D)$, against shelters of different spread as defined by the number of cells m and the probability of hit p, each shelter having a hardness of $x=2$ and $a=4$ subcells

$x=2,a=4$	$p=0.5,m=10$	$p=0.7,m=20$	$p=0.9,m=30$
$E(D) = 0.5$	57	80	94
$E(D) = 0.9$	118	170	197

3. Size of Shelters

If the size of the shelters is reduced, then for a given area, more shelters can be built. As a drops, m will increase. As a case study, let $m=20$, and $a=4$. If shelter size is reduced by half to $a=2$, then for the same area m will double to 40 . If the size is reduced by 25% to $a=3$, then m will be about 27. Figure 10 shows the graph of $E(D)$ vs n for shelters of the three different sizes. Clearly for this case study, using smaller shelters seems to be the most effective way of reducing the vulnerability of shelters.

From this figure the increase in number of shells required to achieve a given level of damage as the hardness of the shelters increases is computed as shown in Table 6.

Table 6: Number of shells required to achieve a given level of expected damage, $E(D)$, against shelters of different size as defined by the number of subcells a, and the number of cells m, each shelter having a hardness of $x=2$. Weapons accuracy is $p=0.7$

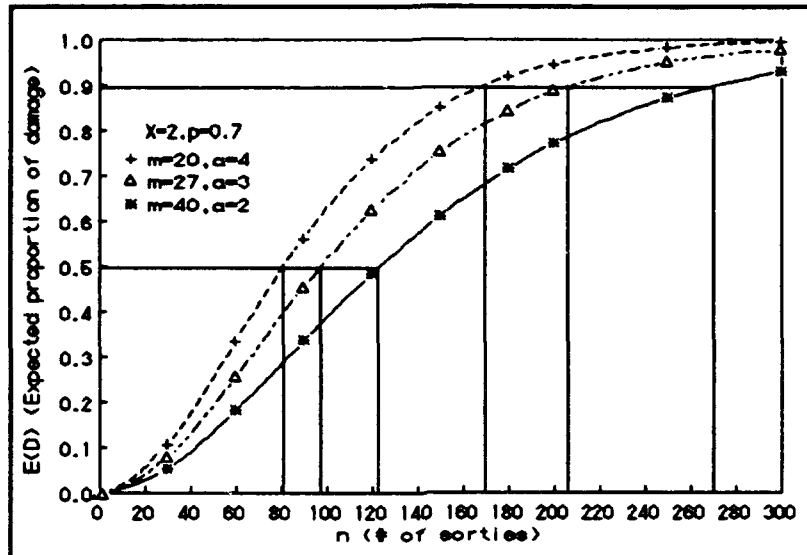| $x=2, p=0.7$	$a=2, m=40$	$a=3, m=27$	$a=4, m=20$
$E(D) = 0.5$	80	98	122
$E(D) = 0.9$	170	206	270

Figure 10: $E(D)$ vs n for shelters of different size

4. Size vs Spread vs Hardness of Shelters

Since there are many parameters to be considered when comparing the options of size, spread and hardness, it is not possible to derive a closed form solution for use in the comparison. As a case study, consider a target of area $m=20$, $p=0.7$ and $a=4$, $x=2$. Three options are:

(1) To harden it to $x=3$, $m=20$, $a=4$, $p=0.7$,

(2) To reduce its size so that $a=2$, $m=40$, $x=2$, $p=0.7$, and

(3) To spread it so that $m=30$, $a=4$, $p=0.9$, $x=2$.

Figure 11 shows $E(D)$ vs n for all three options. Evidently for this case study, hardening the shelter to $x=3$ is a better option with regard to survivability.
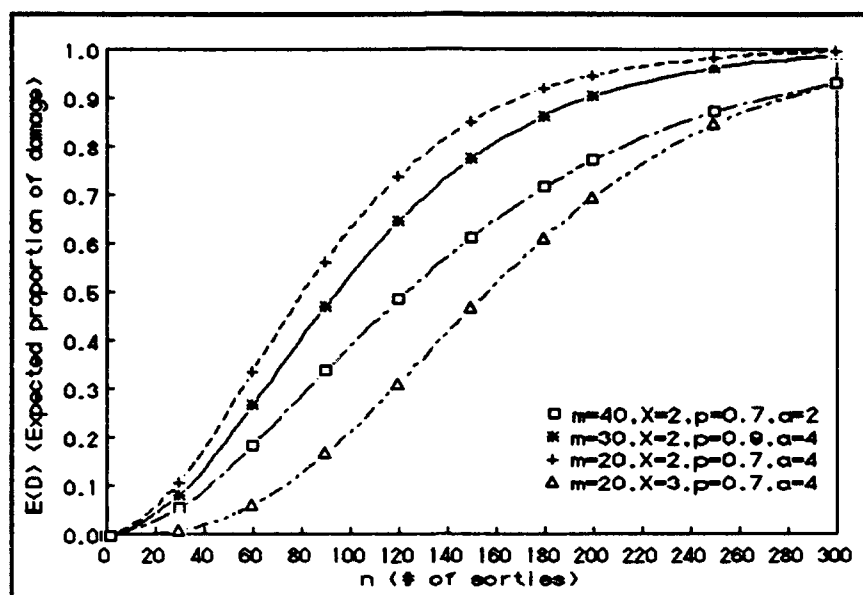
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Figure 11: *E(D)* vs *n* for shelters of different size, spread and hardness

D. ANALYSIS OF ATTACKER'S STRATEGY

For the attacker, the immediate interest is to know how improving accuracy affects performance. Figure 12 and Figure 13 show how improving the accuracy of the weapons will improve expected damage for a given number of shells used.

From these figures the increase in number of shells required to achieve a given level of damage as the accuracy of the weapons increases is computed as shown in Table 7.
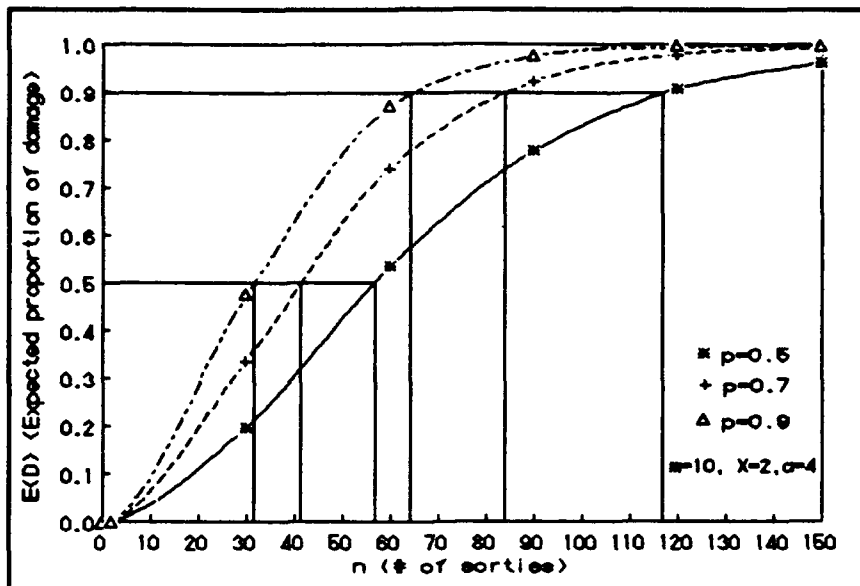
34

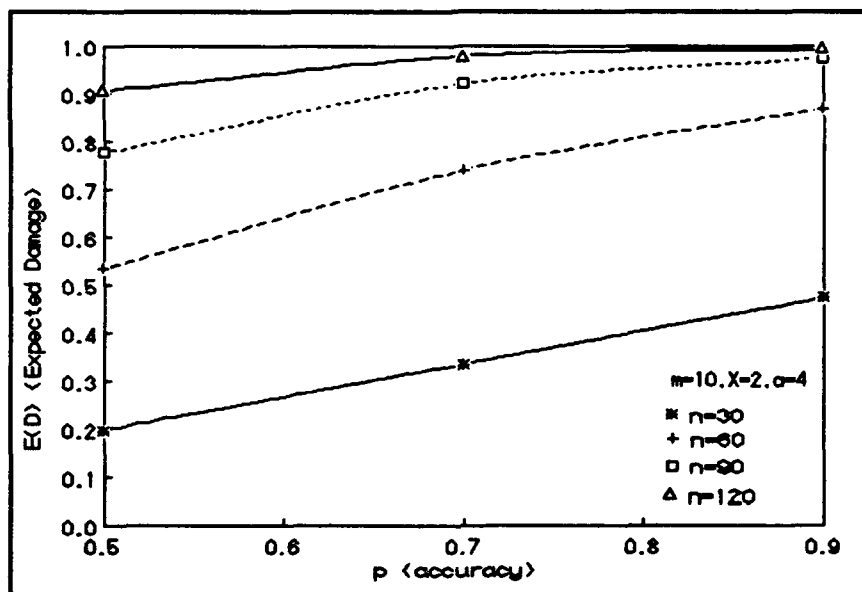Figure 12: $E(D)$ vs n for weapons of different accuracy

Figure 13: $E(D)$ vs p

Table 7:   Number of shells required to achieve a given level
           of expected damage, $E(D)$, when using weapons of
           different accuracy p, against shelters of hardness
           $x=2$, and size $m=10$ and $a=4$

$x=2, m=10, a=4$	$p=0.5$	$p=0.7$	$p=0.9$
$E(D) = 0.5$	31	41	56
$E(D) = 0.9$	65	84	117

E.   SUMMARY OF ANALYSIS

The defender's strategy for improving the survivability of
the shelters includes:

(1) hardening the shelters,

(2) spreading the shelters over a wider area,

(3) using smaller shelters, and

(4) any combination of the above.

The above discussions illustrate how the effectiveness of each
option in terms of the increase in the number of shells
required to achieve a given level of damage may be computed.

The amount of reduction in expected damage as the hardness
of the shelters is increased can be computed as in the
preceding examples.

For the option of spreading the shelter over a larger
area, survivability will only improve if the new p/m is
reduced.

Intuition may suggest that smaller shelters should improve
the survivability of the shelters since the vulnerability of

the shelters is in a sense spread out. The model presented in Scenario 2 provides a means of investigating this possibility.

From the attacker's viewpoint, the effectiveness of increasing weapons accuracy in terms of the reduction in number of shells required to achieve a given level of damage may be readily computed as illustrated in the above discussions.

The next chapter will present a generalized shelter hardness and vulnerability models resulting from the models derived in Chapters III and IV, that can be use for more realistic kill criteria based on engineering analysis of the design of a shelter and the effects of the weapons to be directed against it. In addition, useful conclusions applicable to both models will be discussed.

V DISCUSSIONS

A. GENERALIZED SHELTER HARDNESS AND VULNERABILITY MODELLING

In Scenarios 1 and 2, $E(D)$ and the PK_i can be computed from

$$E(D) = PK_i = \sum_{j=0}^{n} \beta_i(j) \binom{n}{j} \left[\frac{p}{m}\right]^{j} \left[1 - \frac{p}{m}\right]^{n-j},$$

where $\beta_i(j)$ is the probability that shelter i is killed by j hits on the target. For Scenario 1, Kill Criteria 1,

$\beta_i(0) = 0$ and

$\beta_i(j) = 1, \ j \geq 1$

for each shelter $i = 1, \ldots, s$. For Scenario 1, Kill Criteria 2,

$\beta_i(j) = 0, \ j < x$ and

$\beta_i(j) = 1, \ j \geq x$

for each shelter. For Scenario 2, Kill Criteria 1, the $\beta_i(j)$'s are the same as in Scenario 1, Kill Criteria 1. For Scenario 2, Kill Criteria 2, the $\beta_i(j)$'s are computed using Equation (2.6).

The preceding formulation and the examples of its application point to the possibility of introducing other shelter kill criteria based on engineering analysis of the design of a shelter and the effects of the weapons to be

directed against it. The analysis would result in a shelter
hardness and vulnerability distribution $\beta_i(j)$, $j=0,...$ which
could be used in the equation shown above, where as before,
$\beta_i(j)$ is the probability that the ith shelter is killed by j
weapons that hit the shelter.

As an example, one could imagine a kill contour of a
shelter based on engineering analysis like the one shown in
Figure 14.

AREA SURROUNDING SHELTER
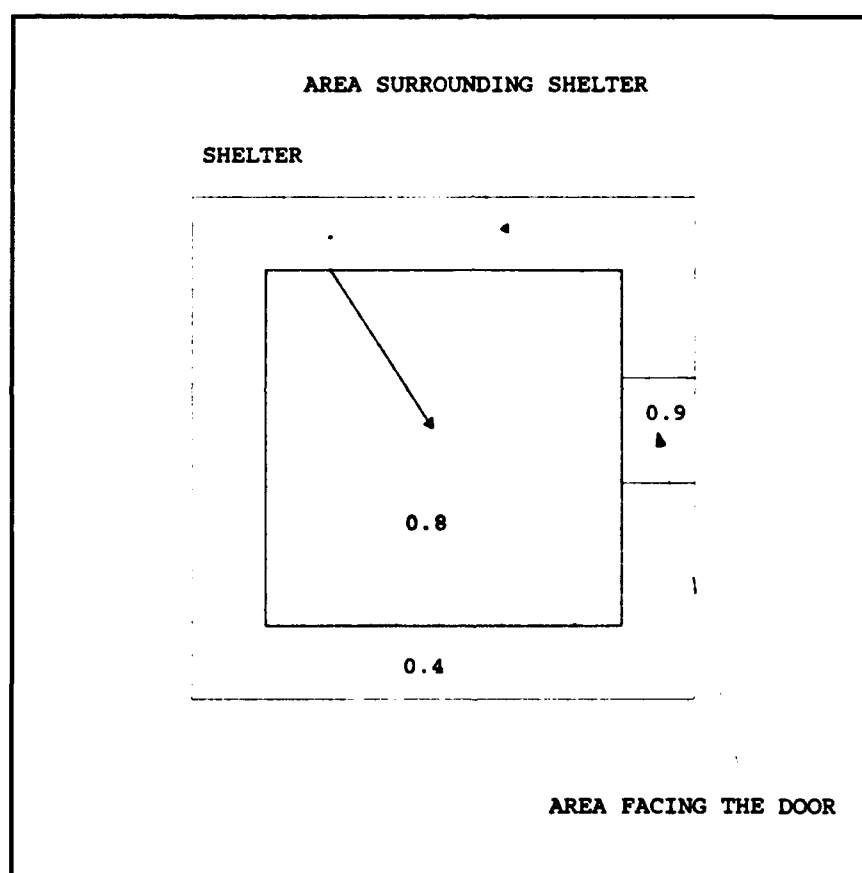
SHELTER

0.9

0.8

0.4

AREA FACING THE DOOR

Figure 14:   Kill contour of a shelter as
provided by engineering analysis

In this case, if the weapon hits the shelter directly, the probability of kill will be 0.8. If the weapon misses the shelter but lands in the near vicinity, the probability of kill will be 0.4 and if the weapon lands on the region facing the door, the probability of kill will be 0.9. Further, assume that the damage mechanism is not cummulative and the proportions of area occupied by the shelter, the area surrounding the shelter and the "door area" are 0.65, 0.25 and 0.05 respectively. Since the weapon is equally likely to hit any region on or near the shelter, then

$$\beta(0) = 0$$

$$\beta(1) = 0.65\text{x}0.8 + 0.25\text{x}0.4 + 0.05\text{x}0.9 = 0.665 = r$$

and $\quad \beta(2) = 1 - (1 - 0.665)^2 = 0.8878$

In general,

$$\beta(j) = 1 - (1 - r)^j.$$

The usefulness of the generalized shelter hardness model lies in the fact that the $\beta(j)$ can be readily modified and computed to reflect more complex kill mechanisms such as cumulative damage due to multiple hits on the same region of the shelter.

B. CONDITIONS UNDER WHICH SHELTER PROBABILITY OF KILL AND DUMP EXPECTED PROPORTION OF DAMAGE COINCIDE

In the models derived for Scenarios 1 and 2, weapons that hit the dump impact uniformly over its target area. The shelters within the dump are all of the same size and have the same kill criteria. These symmetries in the attacks on the shelters lead to the equalities

$$\delta_1(k) = \delta_2(k) = \ldots = \delta_s(k) = \quad \delta(k)$$

between the shelter kill probabilities $\delta_1(k), \delta_2(k), \ldots, \delta_s(k)$ as a result of k weapons that hit the dump, $k=0,1,\ldots,n$, where n is the number of weapons directed at the dump.

Taking the preceding equalities between conditional shelter kill probabilities as a hypothesis, it follows that the proportion of damage to the dump from k hits on the dump is

$$D(k) = \sum_{i=1}^{s} w_i \delta_i(k) = \sum_{i=1}^{s} w_i \delta(k) = \delta(k) \sum_{i=1}^{s} w_i = \delta(k),$$

$k=0,1,\ldots,n$, where $w_1,\ldots,w_s$ are the proportional value weights assigned to the shelters. Then the expected proportion of damage to the dump from the attack is

$$E(D) = \sum_{k=0}^{n} D(k) Pr(N=k) = \sum_{k=0}^{n} \delta(k) Pr(N=k).$$

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where N is the number of hits on the dump.

Also, the probability that the ith shelter is killed by the attack is

$$PK_i = \sum_{k=0}^{n} \delta_i Pr(N=k) = \sum_{k=0}^{n} \delta(k) Pr(N=k).$$

Thus the hypothesis $\delta_1(k) = \delta_2(k) = \ldots = \delta_s(k)$, $k=0,\ldots,n$, leads to the conclusions:

(1) The proportions of the dump damaged by k hits, $D(k)$, $k=0,\ldots,n$ and the expected proportion of the dump damaged by the attack, $E(D)$, do not depend on the value weights w_i, $i=1,\ldots s$, assigned to the shelters, and

(2) The shelter kill probabilities from the attack, PK_i, $i=1,\ldots,s$, are all the same and equal to $E(D)$, the expected proportion of damage to the target.

The computational options offered by the second conclusion are illustrated in the analyses of Scenarios 1 and 2 . For Scenario 1, Kill Criteria 2, $E(D)$ is arrived at by a direct computation of PK_i for a typical shelter. For Scenario 2, Kill

Criteria 2, the PK_i's are arrived at by a direct computation

of $E(D)$, although the alternate option could be exercised.

VI CONCLUSIONS

The methodologies used to analyze Scenarios 1 and 2 can be extended in a variety of ways to cover other scenarios.

Of particular interest is the possibility of incorporating more sophisticated and realistic shelter hardness and vulnerability models as indicated in Chapter V, Section A.

The uniformities in weapons impact and shelter size and vulnerability assumed in Scenarios 1 and 2 have some computationally convenient consequences which are summarized in Chapter V, Section B. However, non-uniform models for weapons impact can be introduced, and shelters of mixed size and hardness can be considered.

LIST OF REFERENCES

[1] J.D.Esary. Studies on Damage Aggregation for Weapons Salvos, Technical Report NPS55-90-16, Naval Postgraduate School, July 1990.

[2] J.D.Esary. Studies on Damage Aggregation for Weapons Salvos II, Technical Report NPSOR-92-007, Naval Postgraduate School, July 1991.

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